Mathematical Representations of the World: Power Law, Tipping Point and Logistic Function

The "Power Law"

Many natural and economic phenomena operate according to a "power law." That is, the relationship between two variables, *y* and *x*, can be written:

$$y = ax^k$$

where *a* and *k* are constants. One of the most familiar power laws is the law of gravity, where the attraction between two bodies is inversely proportional to the square of the distance between them.

In this course we have seen several examples of power law relationships. These include:

- The "Mount Fuji" distribution of natural resources, which says that if we graph resource quality against quantity, we will find a small sharp high value peak, and a broad low-value shoulder.
- The Pareto distribution of wealth, which says 20% of the population own 80% of the wealth and vice versa. The Pareto distribution applies to many social phenomena, such as size of cities and the size of firms.
- "Hyperbolic" discounting—which is the way humans and animals actually discount the future, instead of exponential discounting which is more "logical."
- Unit elasticity demand curve, on which *P*x*Q*=*K*, where *K* is a constant. Alternatively *P*=*K/Q*. Unit elasticity means that spending remains constant, because any price increase is matched by an equivalent drop in quantity.

For more detail, see Wikipedia on power law and Pareto distribution.

The "Tipping Point"

In his best-seller, *The Tipping Point*, Malcolm Gladwell popularized the idea of a point where a very small change can make a huge difference to outcomes. There are many such points in nature and economics. Other related terms include a "phase change" or "discontinuity," or "critical point."

Simple discontinuities are things like cliffs, or edges of lakes and oceans.

Here's a more sophisticated example: Suppose people walk across a field. We want to minimize the damage to the grass. If there are just a few people, they should spread out. But as more and more people cross the field, there comes a point when they will do less damage if they walk single file. A single trampled path is less harmful than trampling scattered all over the field.

The field example translates directly into "succession" of economic land uses. When population density is low, people live on large plots, take their water from wells, and dump their waste in a septic tank (or privy). As density increases, there comes a point when it's more cost-effective to build sewers and water systems. As density increases further, single-family houses switch to townhouses, then to apartment buildings.

Taxes can also have tipping point effects, generally for the worst. An example is the tomato harvester problem, in which a relatively low payroll tax tips the processing tomato industry from hand-picked labor-intensive cultivation to machine-picked

There's a tipping point in the logistic function below: virtually all species have a critical population level such that if population falls below it, the population will continue to decline all the way to extinction.

The Simple Logistic Curve

The simple logistic curve is a handy way to represent not only living populations, but many systems that grow exponentially at first but then approach a limit. For example, a new business may grow rapidly at first, but then converge to a limit as competitors move in. At low rates of growth, the logistic formula gives a characteristic "S" function. At higher rates of growth, the logistic function can produce oscillating patterns; at even higher rates it goes chaotic. See James Glieck, *Chaos: Making a New Science*, Viking, 1987, chapter on "Life's Ups and Downs." Also see Wikipedia on logistic function.

Let:

N_t = population at time *t*,

K = carrying capacity or maximum long-run population

B = base level, below which population cannot survive

r = growth factor

Then: The first equation gives the **population** at time *t+1* as a function of population at time *t*. The second equation gives the **growth** or change in population from *t* to *t+1*.

$$N_{t+1} = N_t + r(N_t - B)\left[1 - \frac{N_t - B}{K - B}\right] \ldots or$$

$$\Delta N = N_{t+1} - N_t = r(N_t - B)\left[1 - \frac{N_t - B}{K - B}\right]$$

Where population is close to the base level, growth is approximately exponential, starting from base B, that is:

$$N_{t+1} = N_t + r(N_t - B)\ldots or$$
$$N_{t+1} - B = (N_t - B)(1+r)\ldots or$$
$$\Delta N = (N_{t+1} - B) - (N_t - B) = r(N_t - B)$$

As population approaches *K*, growth falls to zero.

Setting the first derivative of N_t equal to zero shows that growth is maximum halfway between B and K, or at

$$N_{g\max} = \frac{K+B}{2}$$

This is also the **inflection point** in the population curve, that is the point at which the curve switches from an increasing rate of growth to a decreasing rate.

If the initial value of N_t is greater than *K*, growth is negative, and population falls back to *K*.

If the initial value of N_t is less than *B*, growth is negative, and population falls to zero.

Point B is an unstable equilibrium, a "tipping point," where the slightest deviation in either direction leads either to population growth or population collapse. It is a good way to represent the fact that once populations of most species fall below a certain level, the species will inevitably go extinct. For example some species, like the passenger pigeon, breed communally. Below a certain level, they cannot breed successfully.

Notice that the **growth factor**, *r*, is not the same as **growth**:

$$\Delta N = N_{t+1} - N_t = r(N_t - B)\left[1 - \frac{N_t - B}{K - B}\right]$$

Growth is also not the same as **growth rate**, which is growth divided by the population above base *B*, or

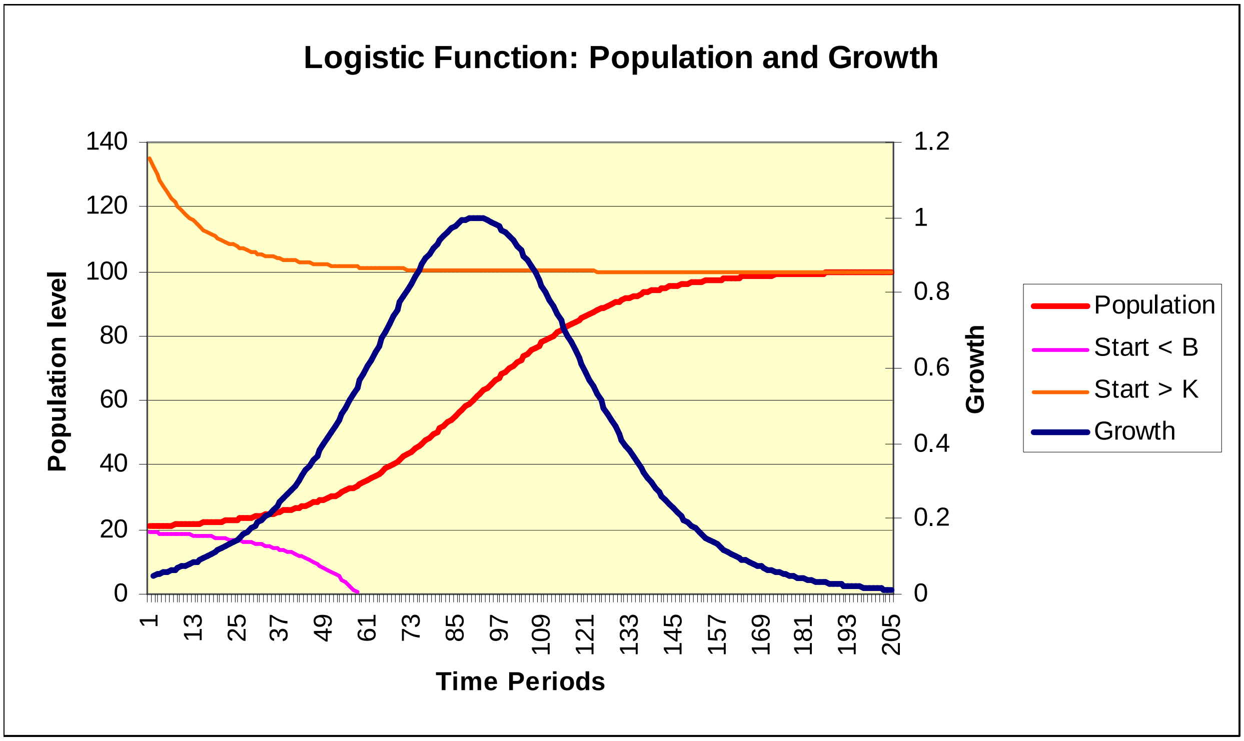
$$Growth\_rate = r\left[1 - \frac{N_t - B}{K - B}\right]$$

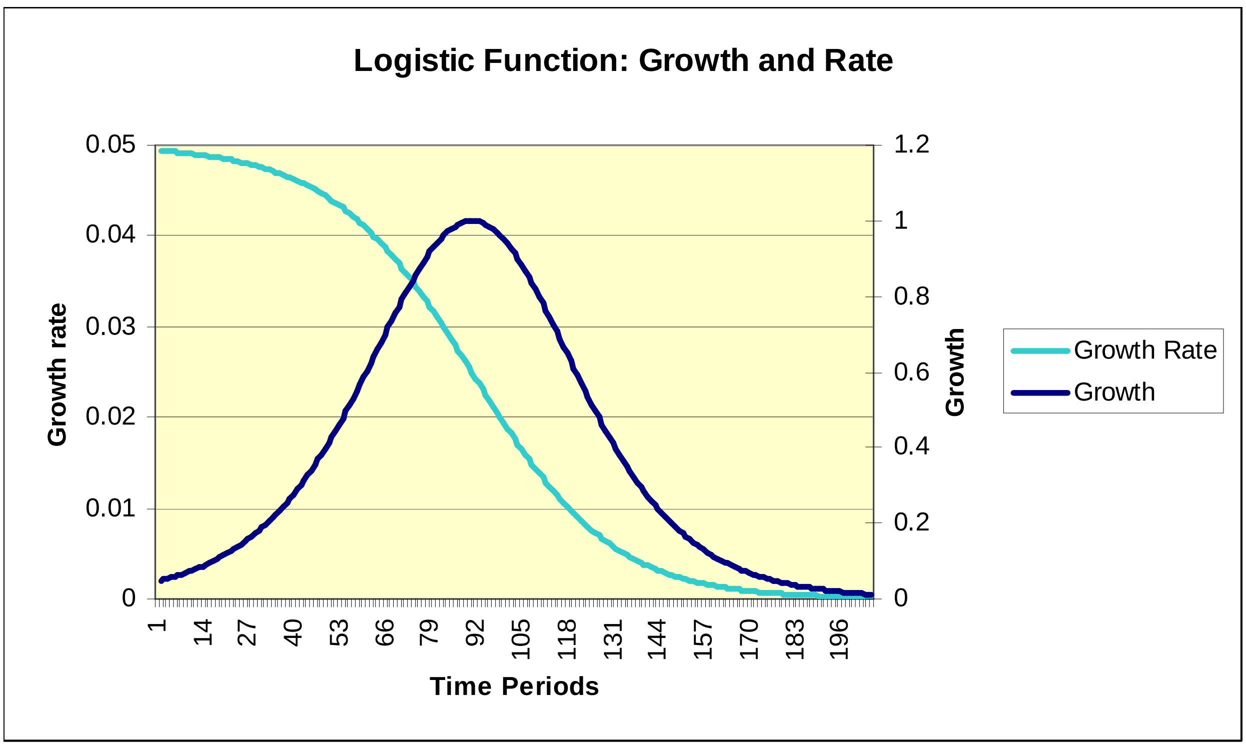
The **maximum sustainable yield** or **MSY** of a population is the maximum that can be harvested without sending the population into decline. It occurs at the peak of the growth curve, at $N_{gmax} = (K+B)/2$, shown above. From this it follows that the MSY formula is:

$$MSY = r\frac{K-B}{4}$$

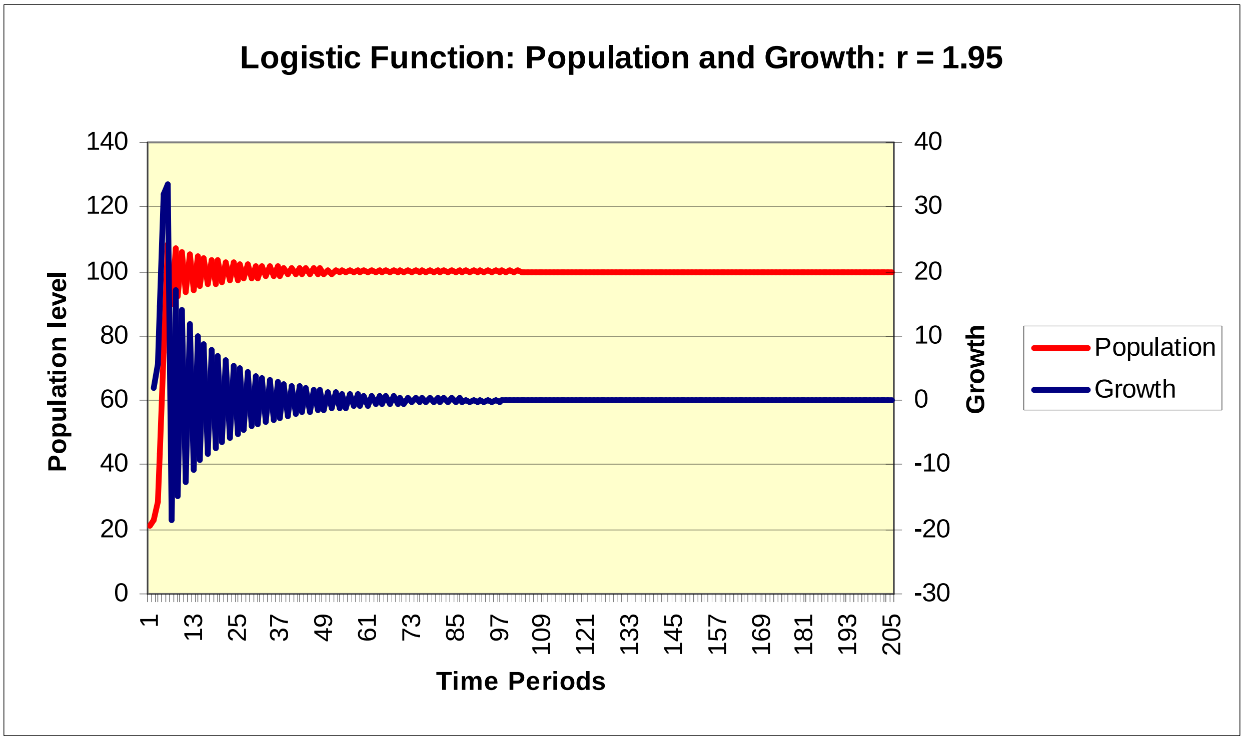
If the MSY is harvested when the population is above the maximum growth point, $(K+B)/2$, the population will decline until it reaches the maximum growth point, where population will stabilize. However, if MSY is harvested when population is below the maximum growth point, the population will continue to decline. Unless harvests are reduced, the population will eventually get below B, and head for extinction even with no further harvesting. Thus $N_{gmax} = (K+B)/2$ under MSY harvesting is itself another tipping point or unstable equilibrium, where the tiniest excess can send the population into decline.

Here are graphs of a logistic function for $K = 100$, $B = 20$ and $r = .05$. The first graph shows population for three starting points: slightly above B, red line; slightly below B, pink line; and above K, orange line. For these parameters, the point of maximum growth occurs at population of 60, halfway between $B = 20$ and $K = 100$. The MSY = $r(K-B)/4 = 1$. See Cleveland Logistic Function.xls

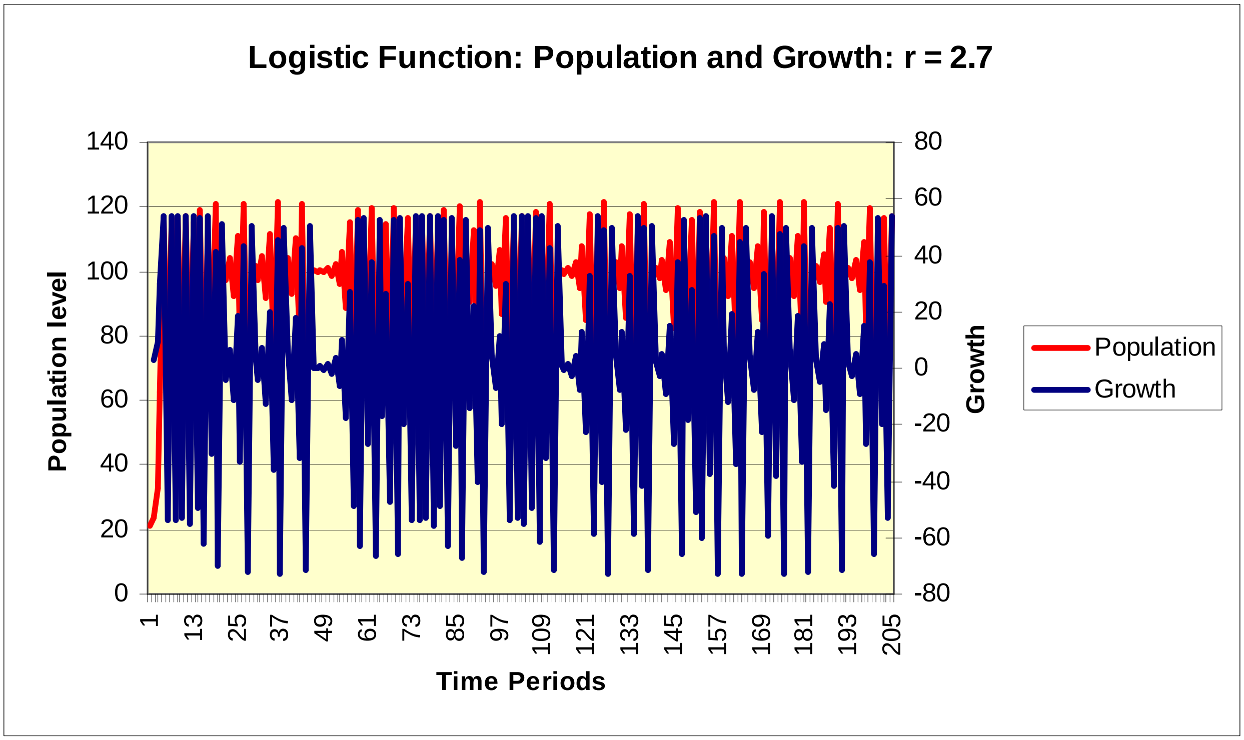


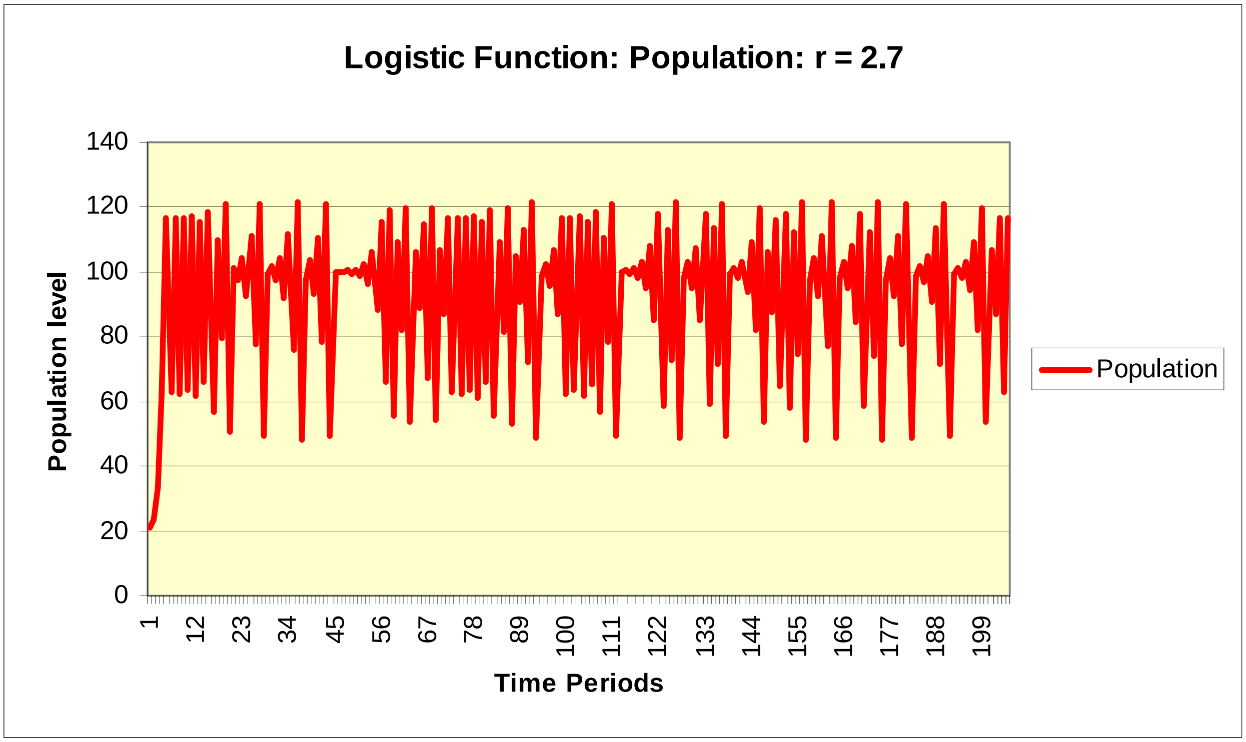


If we increase the growth factor, *r*, the logistic function rises to *K* faster and faster. Then around *r* = 1.2, it starts to oscillate before settling down. As r rises, the oscillations get more pronounced. The next graph shows population and growth for *r* = 1.95.



Eventually, population and growth go chaotic. The next graphs shows patterns for r = 2.7. Some fast-growing populations such as lemmings seem to behave chaotically.





In a population with a high growth factor, *r*, it can be difficult to identify either carrying capacity, *K*, or maximum sustainable yield, MSY. It can also be difficult to determine whether precipitous population declines are due to the naturally chaotic dynamics of the population or to over-harvesting or to other human impact on related ecosystems. That's what makes ecology so exciting!